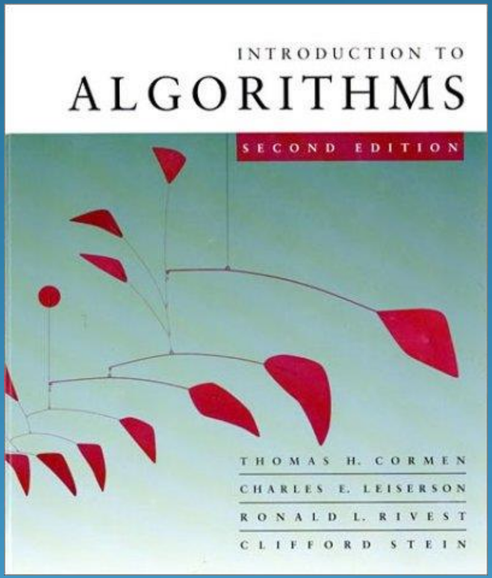
Introduction to Algorithms Design &Analysis



Lecture 1

Prof. Sushree Sangita Jena

Design and Analysis of Algorithms

- ***Analysis:*** predict the cost of an algorithm in terms of resources and performance

- ***Design:*** design algorithms which minimize the cost

The problem of sorting

Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Example:

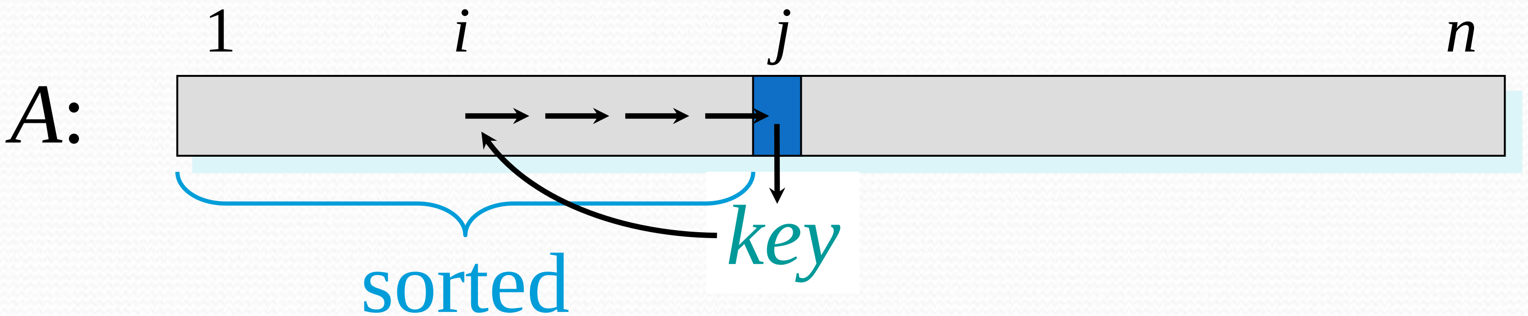
Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9

Insertion sort

"pseudocode"

```
INSERTION-SORT (A, n)        ⊳ A[1 . . n]
    for j ← 2 to n
        do key ← A[ j]
           i ← j – 1
           while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
           A[i+1] = key
```



Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

2 4 8 9 3 6

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

2 4 8 9 3 6

2 4 8 9 3 6

2 3 4 8 9 6

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6
2	3	4	8	9	6

Example of insertion sort

8	2	4	9	3	6	
2	8	4	9	3	6	
2	4	8	9	3	6	
2	4	8	9	3	6	
2	3	4	8	9	6	
2	3	4	6	8	9	*done*

Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Major Simplifying Convention: Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
    - $T_A(n) =$ time of A on length n inputs
- Generally, we seek upper bounds on the running time, to have a guarantee of performance.

Kinds of analyses

Worst-case: (usually)

- $T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)

- $T(n)$ = expected time of algorithm over all inputs of size n.
- Need assumption of statistical distribution of inputs.

Best-case: (NEVER)

- Cheat with a slow algorithm that works fast on *some* input.

Machine-independent time

What is insertion sort's worst-case time?

BIG IDEAS:

- ***Ignore machine dependent constants***, otherwise impossible to verify and to compare algorithms
- Look at ***growth*** of $T(n)$ as $n \to \infty$.

"Asymptotic Analysis"

Θ-notation

DEF:

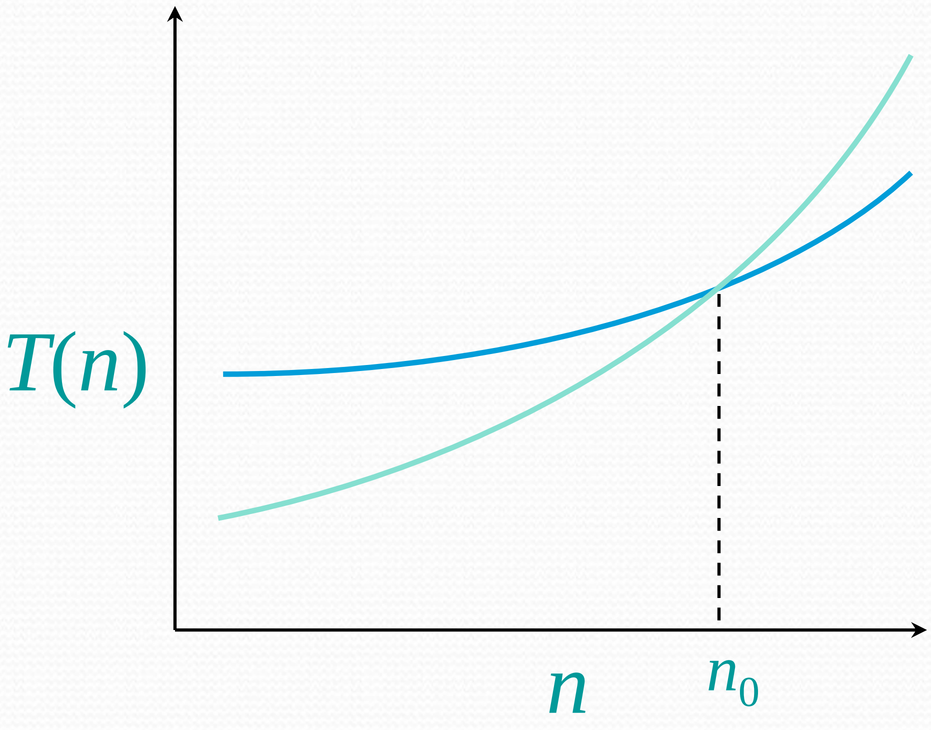
$\Theta(g(n)) = \{ f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0$ $\}$

Basic manipulations:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- Asymptotic analysis is a useful tool to help to structure our thinking toward better algorithm
- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing

Insertion sort analysis

Worst case: Input reverse sorted.

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta\left(n^2\right)$$ [arithmetic series]

Average case: All permutations equally likely.

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \Theta\left(n^2\right)$$

Is insertion sort a fast sorting algorithm?

- Moderately so, for small n.
- Not at all, for large n.

Example 2: Integer Multiplication

- Let X = A B and Y = C D where A,B,C and D are n/2 bit integers
- Simple Method: $XY = (2^{n/2}A+B)(2^{n/2}C+D)$
- Running Time Recurrence

  $$T(n) < 4T(n/2) + 100n$$

- Solution $T(n) = \theta(n^2)$

Better Integer Multiplication

- Let $X = \boxed{A \mid B}$ and $Y = \boxed{C \mid D}$ where A,B,C and D are n/2 bit integers
- Karatsuba:

  $$XY = (2^{n/2}+2^{n})AC+2^{n/2}(A-B)(C-D) + (2^{n/2}+1)\,BD$$

- Running Time Recurrence

  $$T(n) < 3T(n/2) + 100n$$

- Solution: $\theta(n) = O(n^{\log 3})$

Example 3:Merge sort

MERGE-SORT $A[1 \,.\,.\, n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**

Merging two sorted arrays

20	12
13	11
7	9
2	1

Merging two sorted arrays

20	12
13	11
7	9
2	1

1

Merging two sorted arrays

20	12		20	12
13	11		13	11
7	9		7	9
2	1		2	

1

Merging two sorted arrays

20	12		20	12
13	11		13	11
7	9		7	9
2	1		2	
	1			2

Merging two sorted arrays

20	12	20	12	20	12
13	11	13	11	13	11
7	9	7	9	7	9
2	1	2			
1		2			

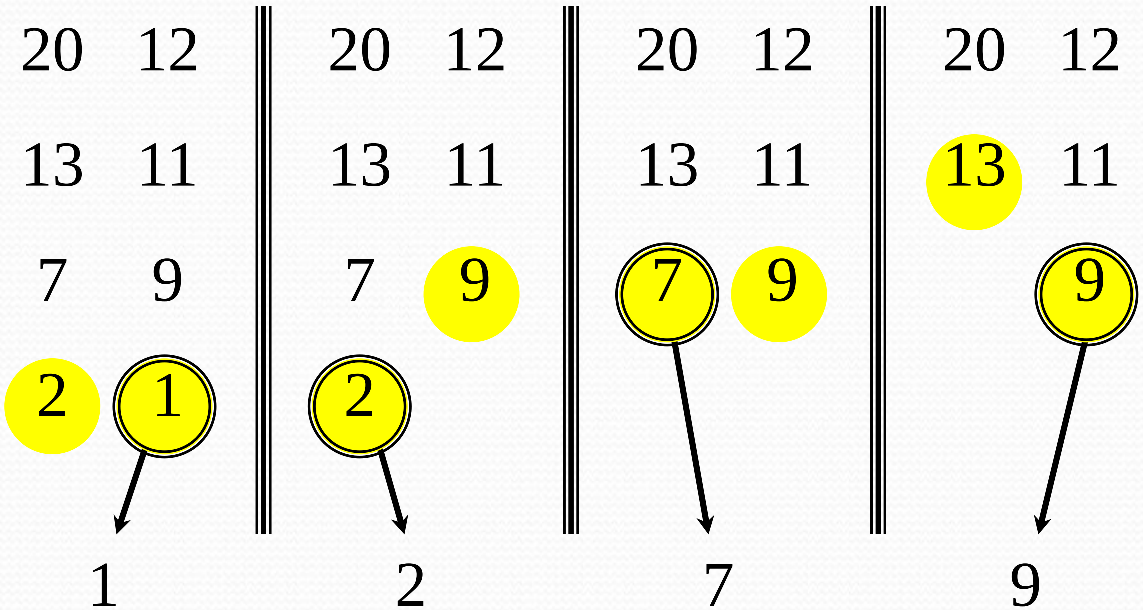
Merging two sorted arrays

20	12		20	12		20	12
13	11		13	11		13	11
7	9		7	9		7	9
2	1		2				
1			2			7	

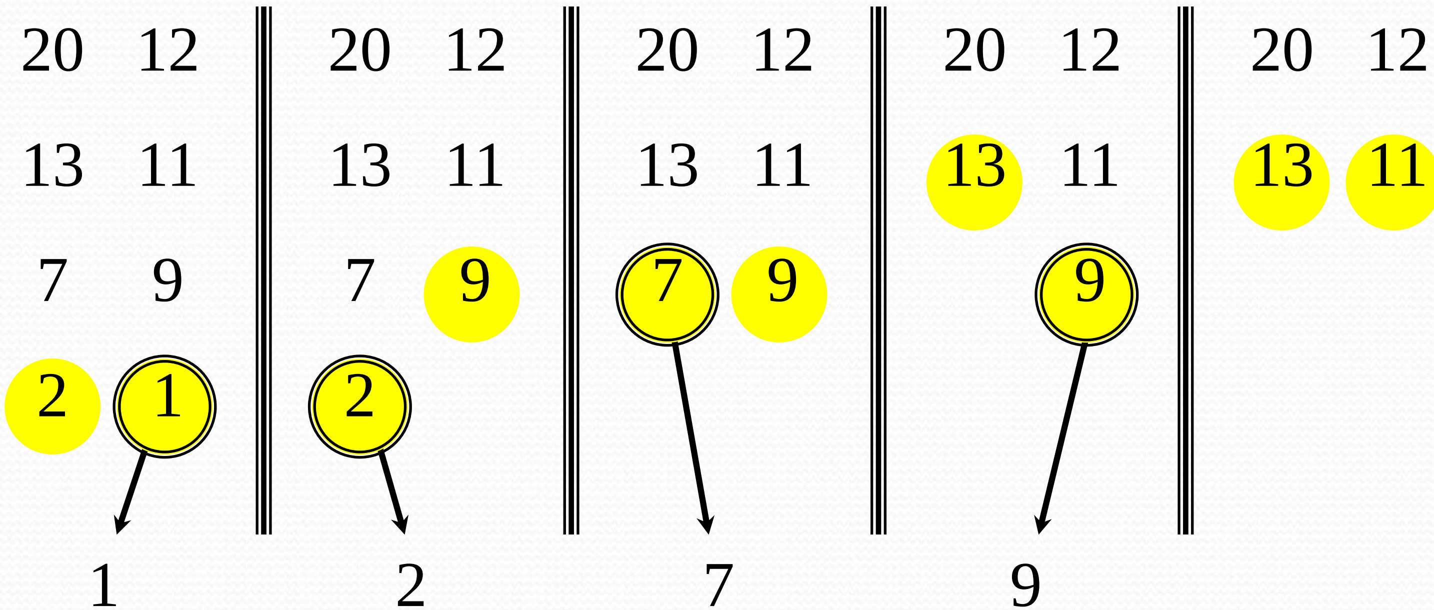
Merging two sorted arrays

20	12	20	12	20	12	20	12
13	11	13	11	13	11	13	11
7	9	7	9	7	9		9
2	1	2					
1		2		7			

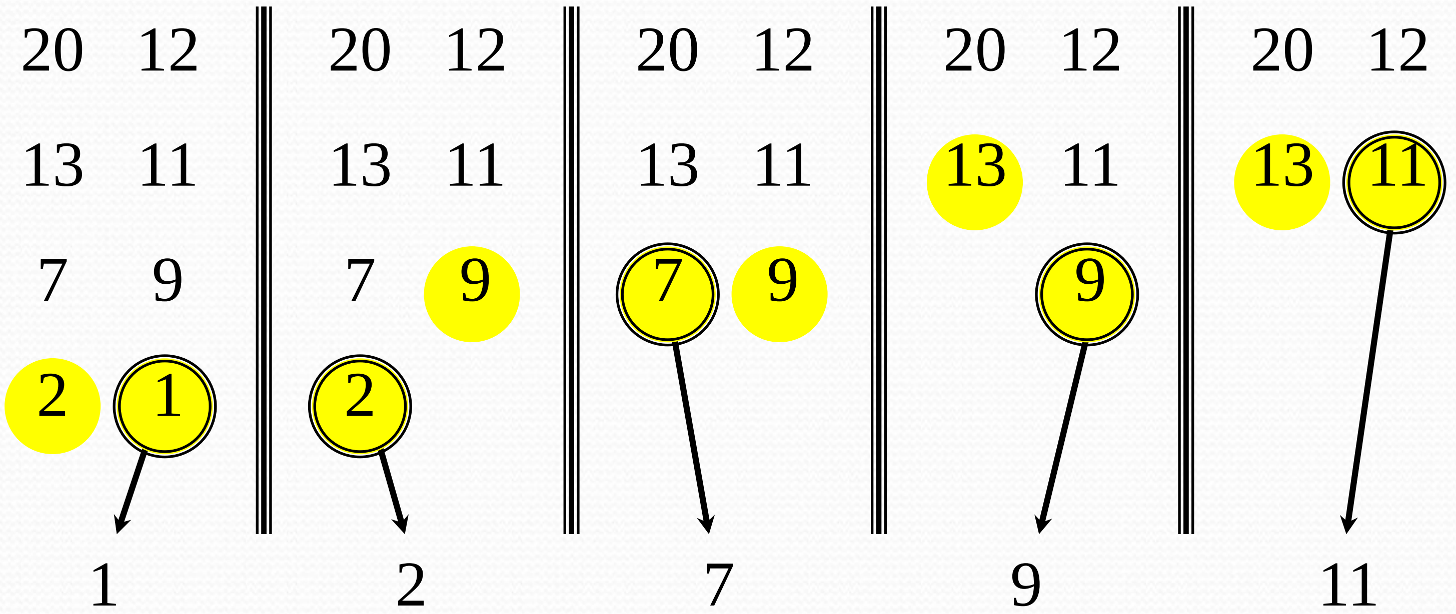
Merging two sorted arrays



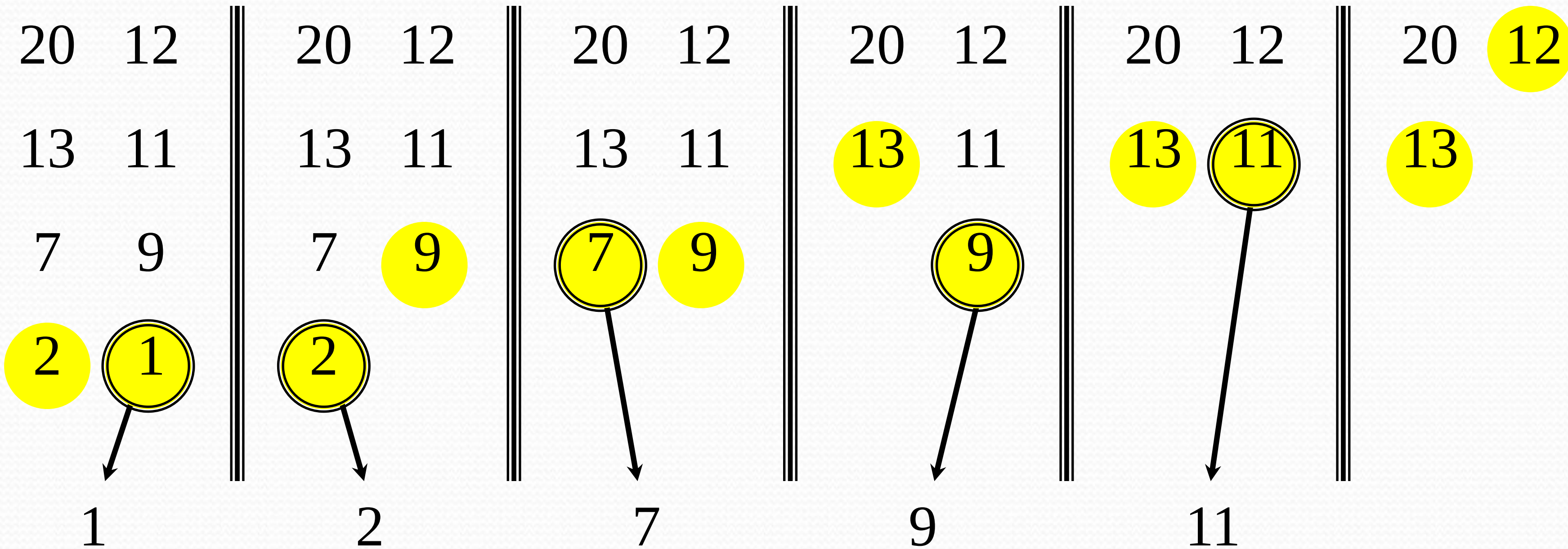
Merging two sorted arrays



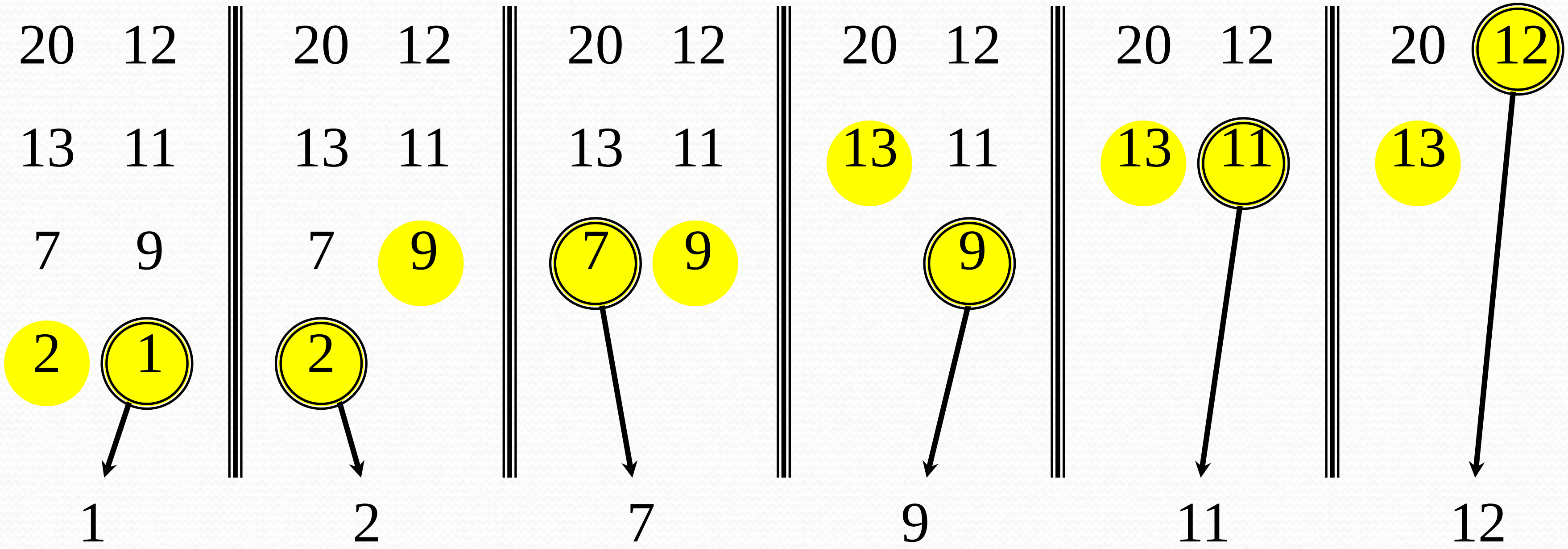
Merging two sorted arrays



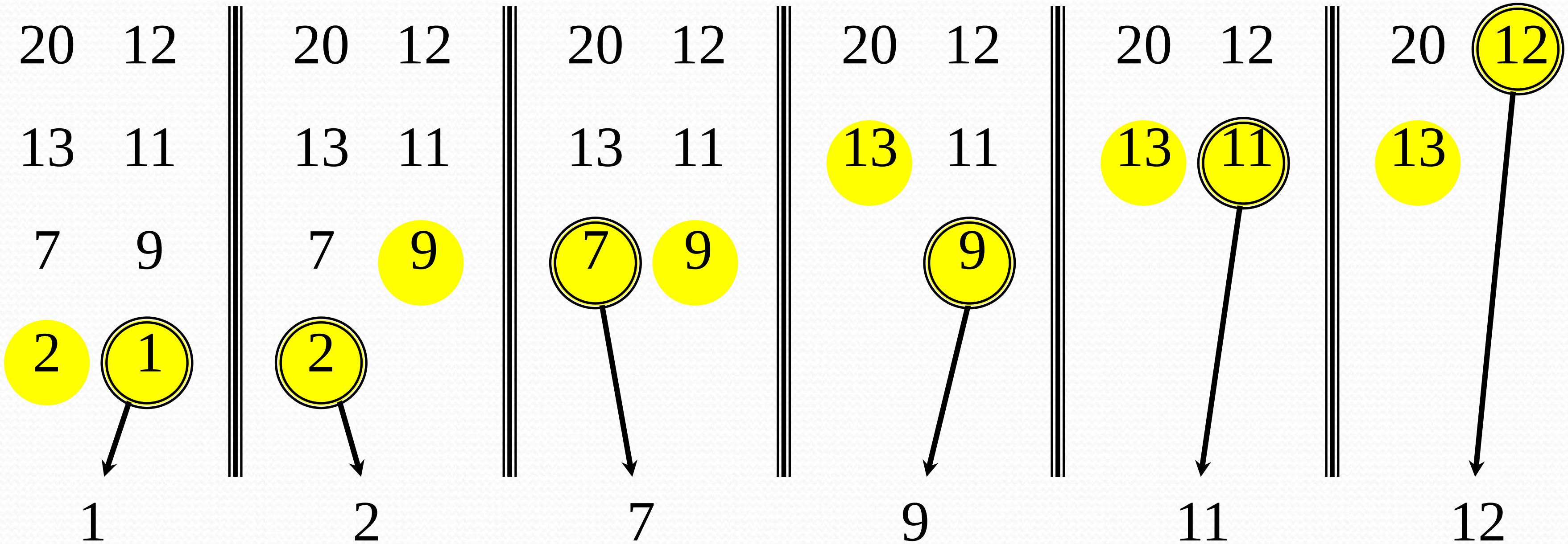
Merging two sorted arrays



Merging two sorted arrays



Merging two sorted arrays



Time = $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

$T(n)$ | **MERGE-SORT** $A[1\ ..\ n]$

$\Theta(1)$ | 1. If $n = 1$, done.

$2T(n/2)$ | 2. Recursively sort $A[\,1\ ..\ \lceil n/2\rceil\,]$ and $A[\,\lceil n/2\rceil+1\ ..\ n\,]$.

$\Theta(n)$ | ***3. "Merge"*** the 2 sorted lists

Sloppiness: Should be $T(\lceil n/2\rceil) + T(\lfloor n/2\rfloor)$, but it turns out not to matter asymptotically.

Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- Lecture 2 provides several ways to find a good upper bound on $T(n)$.

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$T(n)$$

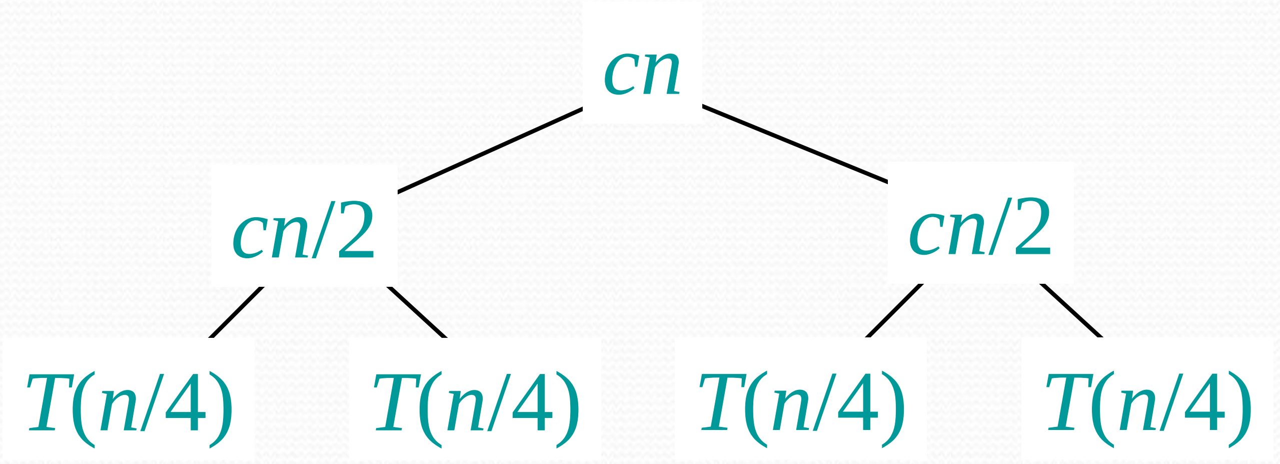
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$
\begin{array}{ccc}
 & cn & \\
T(n/2) & & T(n/2)
\end{array}
$$

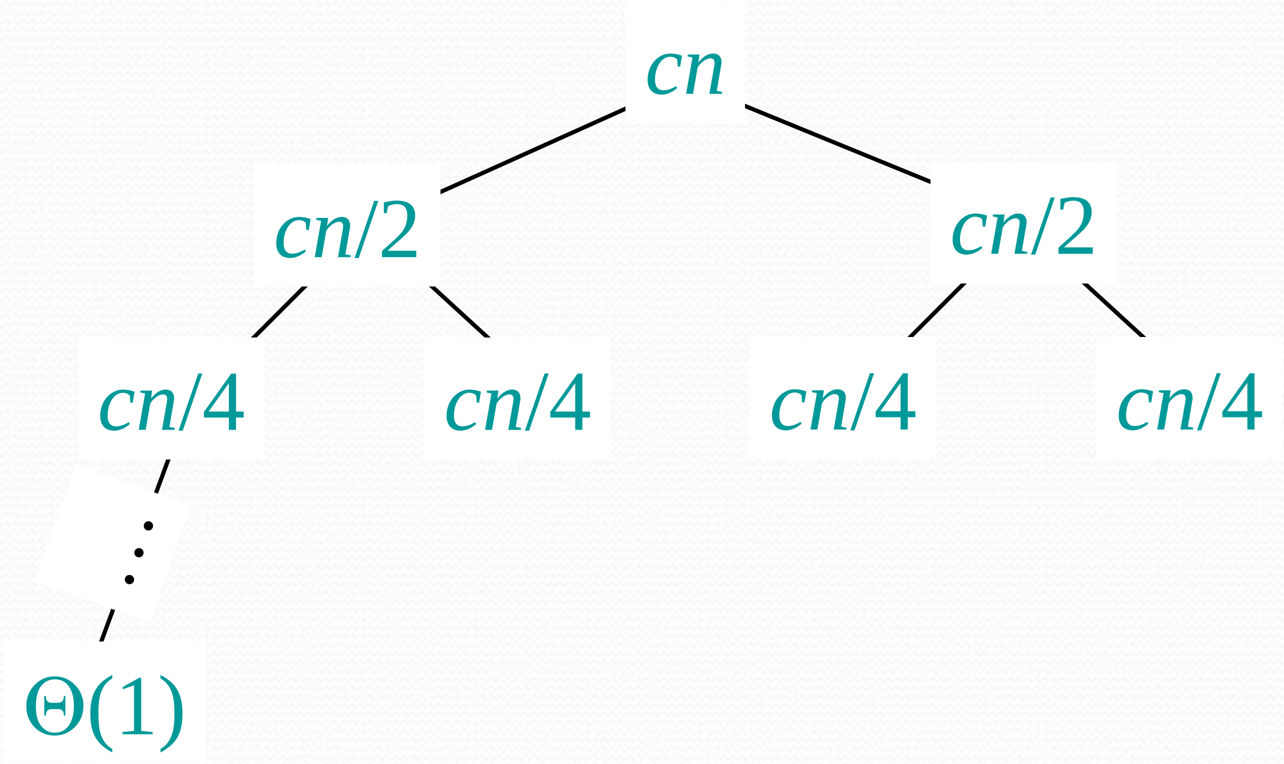
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



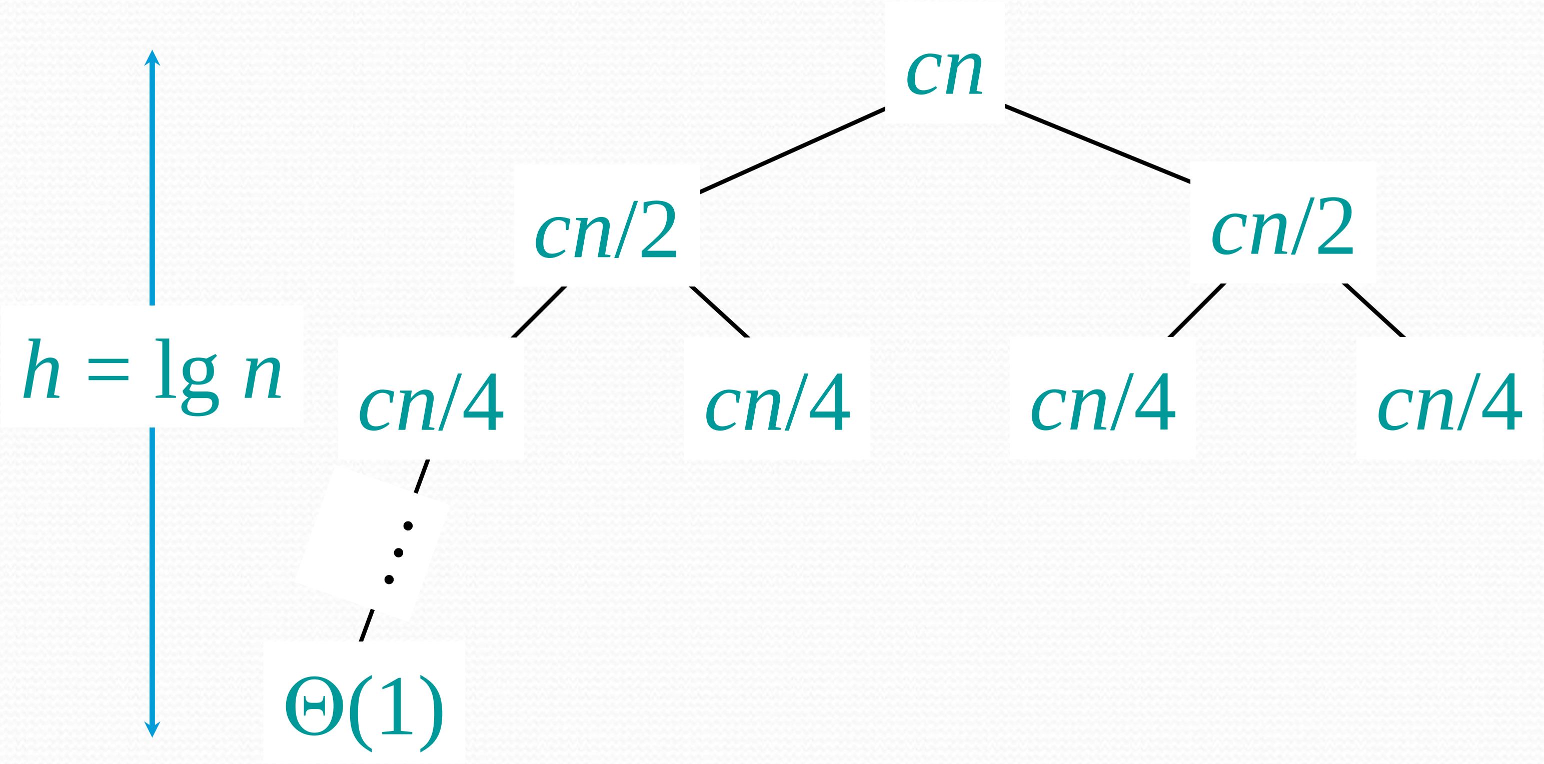
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



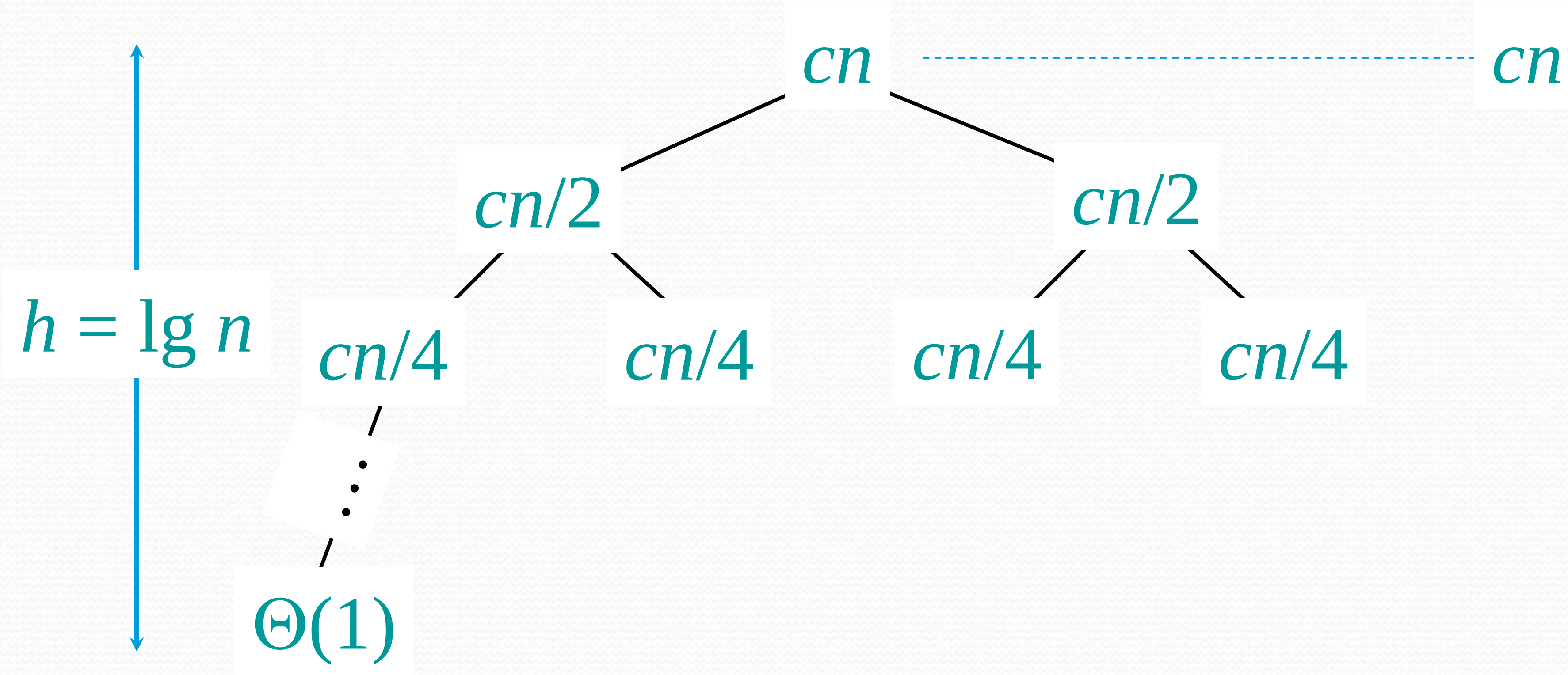
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



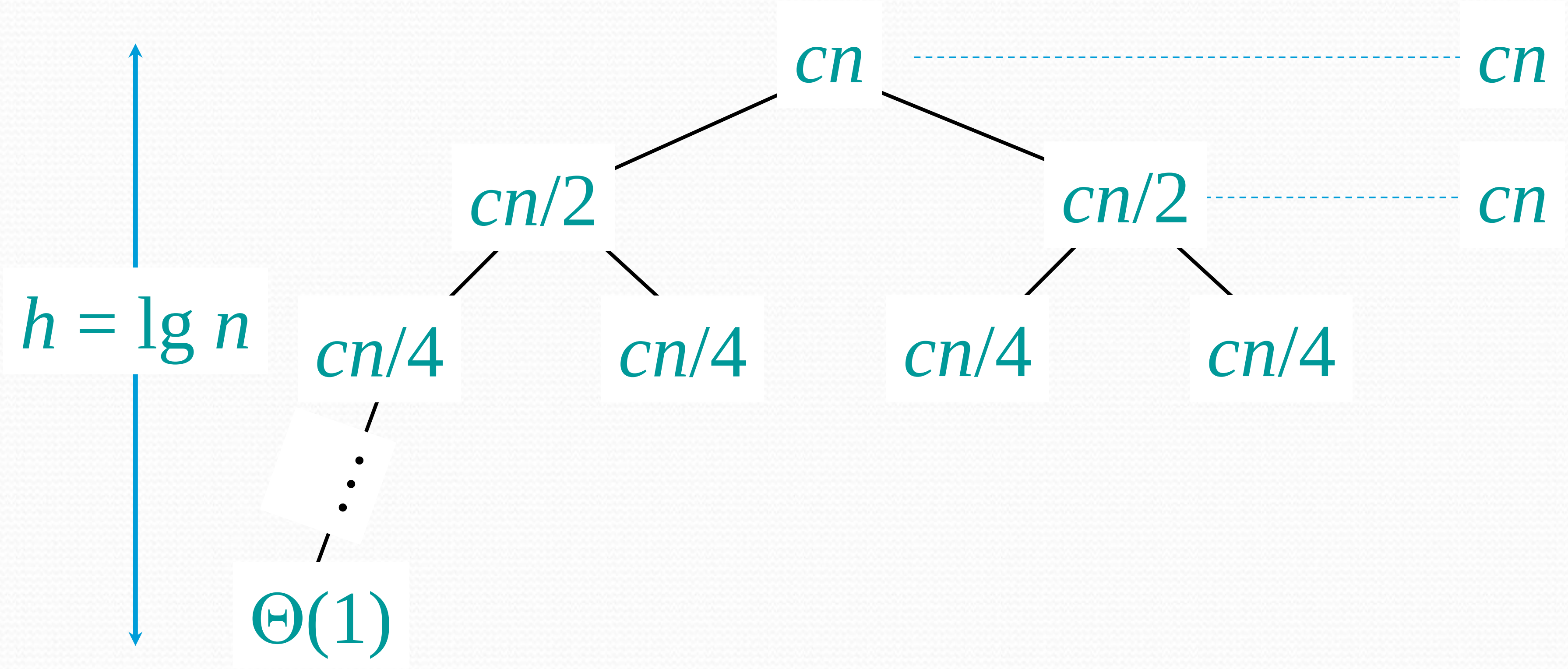
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



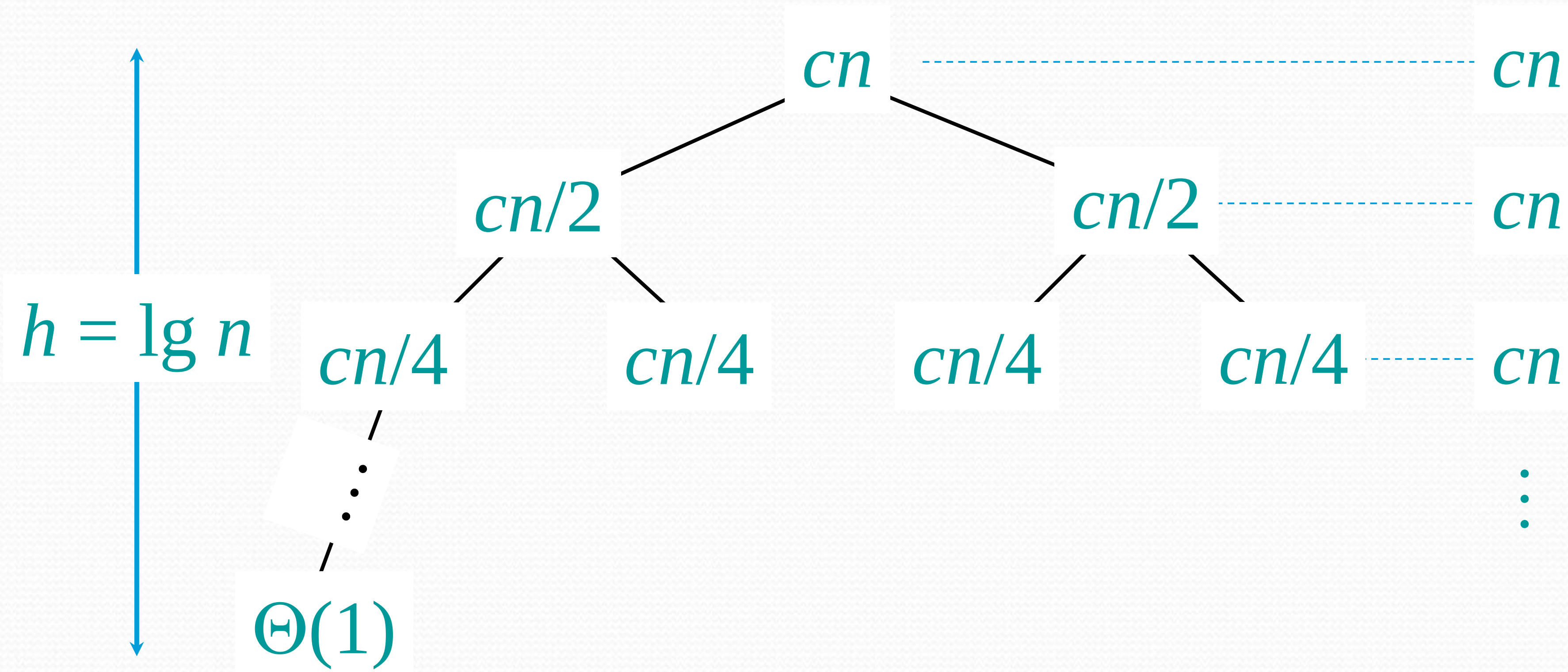
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



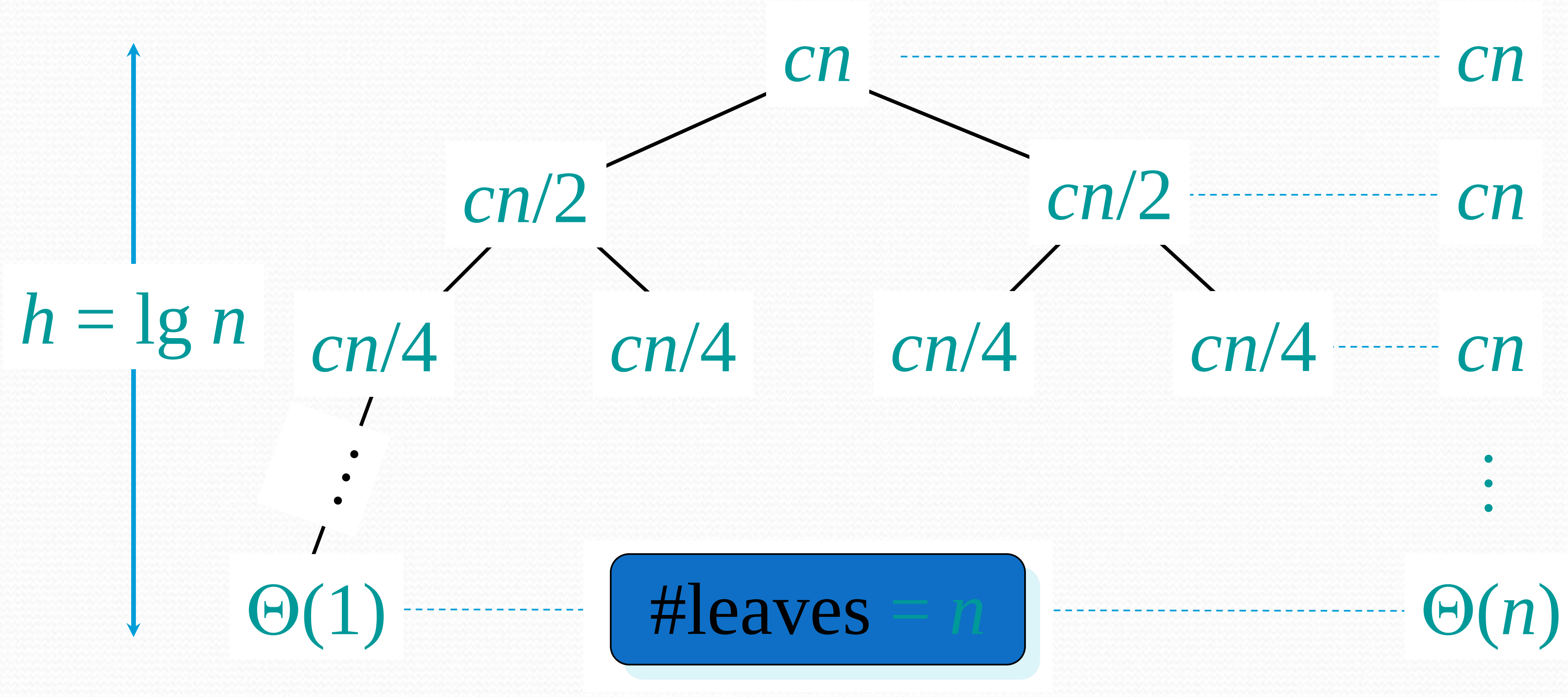
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



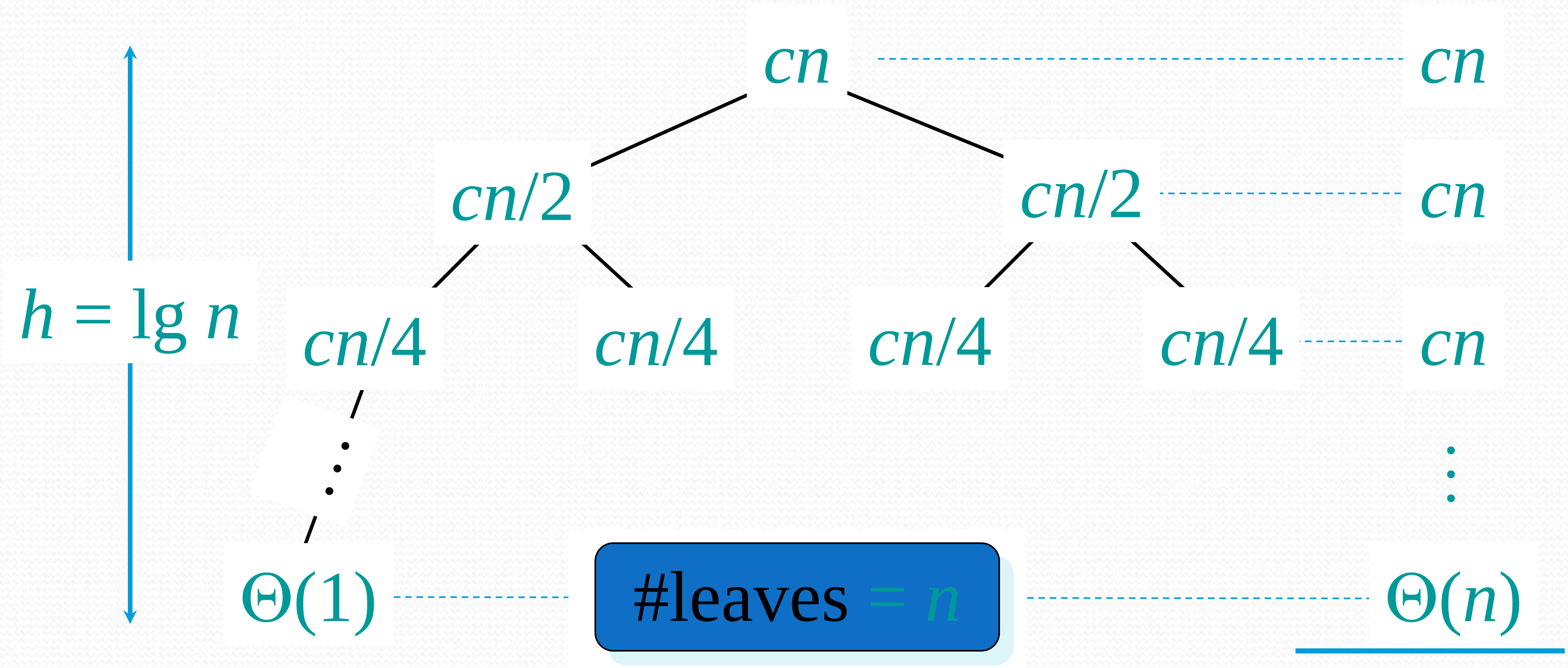
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



Total $= \Theta(n \lg n)$

Conclusions

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.