The Z-Transform

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z-Transform

- The z-transform is the most general concept for the transformation of discrete-time series.
- The Laplace transform is the more general concept for the transformation of continuous time processes.
- For example, the Laplace transform allows you to transform a differential equation, and its corresponding initial and boundary value problems, into a space in which the equation can be solved by ordinary algebra.
- The switching of spaces to transform calculus problems into algebraic operations on transforms is called operational calculus. The Laplace and z transforms are the most important methods for this purpose.

The Transforms

The Laplace transform of a function *f*(*t*):

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

The one-sided z-transform of a function *x*(*n*):

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

The two-sided z-transform of a function *x*(*n*):

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Relationship to Fourier Transform

Note that expressing the complex variable *z* in polar form reveals the relationship to the Fourier transform:

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{i\omega})^{-n}, \text{ or}$$

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-i\omega n}, \text{ and if } r = 1,$$

$$X(e^{i\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$$

which is the Fourier transform of $x(n)$.

Region of Convergence

The z-transform of *x*(*n*) can be viewed as the Fourier transform of *x*(*n*) multiplied by an exponential sequence r^{-n}, and the z-transform may converge even when the Fourier transform does not.

By redefining convergence, it is possible that the Fourier transform may converge when the z-transform does not.

For the Fourier transform to converge, the sequence must have finite energy, or:

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

Convergence, continued

The power series for the z-transform is called a Laurent series:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The Laurent series, and therefore the z-transform, represents an analytic function at every point inside the region of convergence, and therefore the z-transform and all its derivatives must be continuous functions of *z* inside the region of convergence.

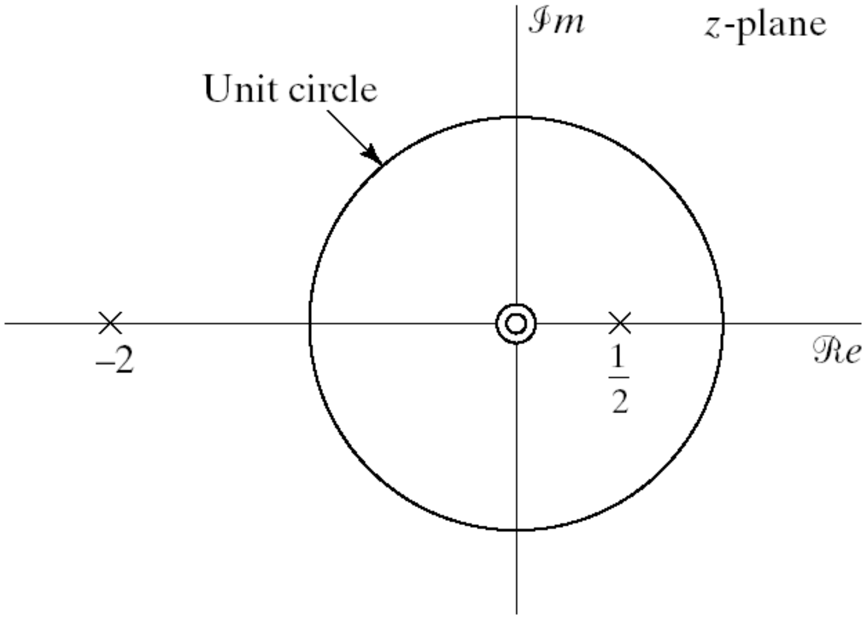
In general, the Laurent series will converge in an annular region of the z-plane.

Properties of The ROC of Z-Transform

- The ROC is a ring or disk centered at the origin
- DTFT exists if and only if the ROC includes the unit circle
- The ROC cannot contain any poles
- The ROC for finite-length sequence is the entire z-plane
  - except possibly z=0 and z=∞
- The ROC for a right-handed sequence extends outward from the outermost pole possibly including z= ∞
- The ROC for a left-handed sequence extends inward from the innermost pole possibly including z=0
- The ROC of a two-sided sequence is a ring bounded by poles
- The ROC must be a connected region
- A z-transform does not uniquely determine a sequence without specifying the ROC

Stability, Causality, and the ROC

- Consider a system with impulse response h[n]
- The z-transform H(z) and the pole-zero plot shown below
- Without any other information h[n] is not uniquely determined
  - $|z|>2$ or $|z|<½$ or $½<|z|<2$
- If system stable ROC must include unit-circle: $½<|z|<2$
- If system is causal



Poles and Zeros

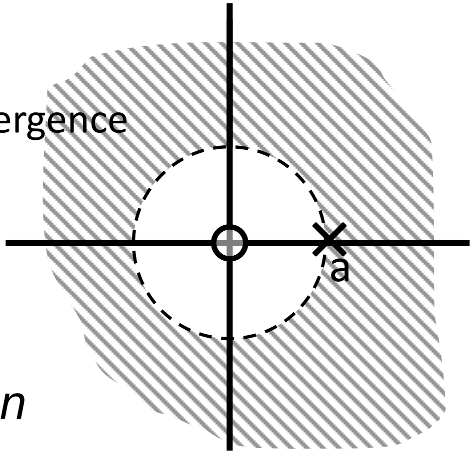
When *X*(*z*) is a rational function, i.e., a ration of polynomials in *z*, then:

1. **The roots of the numerator polynomial are referred to as the zeros of *X*(*z*), and**
2. **The roots of the denominator polynomial are referred to as the poles of *X*(*z*).**

Note that no poles of *X*(*z*) can occur within the region of convergence since the z-transform does not converge at a pole.

Furthermore, the region of convergence is bounded by poles.

Example



$$x(n) = a^n u(n)$$

The z-transform is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Which converges to:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad for \quad |z| > |a|$$

Clearly, *X*(*z*) has a zero at *z* = 0 and a pole at *z* = *a*.

Inverse z-Transform

The inverse z-transform can be derived by using Cauchy's integral theorem. Start with the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Multiply both sides by z^{k-1} and integrate with a contour integral for which the contour of integration encloses the origin and lies entirely within the region of convergence of $X(z)$:

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = \frac{1}{2\pi i}\oint_C \sum_{n=-\infty}^{\infty} x(n) z^{-n+k-1} dz$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi i}\oint_C z^{-n+k-1} dz$$

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = x(n) \text{ is the inverse z - transform.}$$

Properties

- z-transforms are linear:

$$\mathcal{Z}[ax(n)+by(n)] = aX(z)+bY(z)$$

- The transform of a shifted sequence:

$$\mathcal{Z}[x(n+n_0)] = z^{n_0}X(z)$$

- Multiplication:

$$\mathcal{Z}\lfloor a^n x(n) \rfloor = Z(a^{-1}z)$$

But multiplication will affect the region of convergence and all the pole-zero locations will be scaled by a factor of a.

BIBO Stability

- Rule #1: Poles inside unit circle (causal signals)
- Rule #2: Unit circle in region of convergence

  Analogy in continuous-time: imaginary axis would be in region of convergence of Laplace transform

- Example: $a^n u[n] \overset{z}{\leftrightarrow} \frac{1}{1 - a\, z^{-1}} \quad \text{for} \quad |z| > |a|$

  BIBO stable if $|a| < 1$ by rule #1

  BIBO stable if $|z| > |a|$ includes unit circle; hence, $|a| < 1$ by rule #2

BIBO means Bounded-Input Bounded-Output

Inverse *z*-transform

- Definition $$h[n] = \frac{1}{2\pi\ j} \oint_R H(z)\, z^{-n+1} dz$$
- Using the definition requires a contour integration in the complex *z*-plane
  - Use Cauchy residue theorem (from complex analysis) OR
  - Use transform tables and transform pairs?
- Fortunately, we tend to be interested in only a few basic signals (pulse, step, etc.)
  - Virtually all signals can be built from these basic signals
  - For common signals, *z*-transform pairs have been tabulated

Example

$$X(z) = \frac{z^2 + 2z + 1}{z^2 - \frac{3}{2}z + \frac{1}{2}}$$

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}$$

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

- Ratio of polynomial z-domain functions
- Divide through by the highest power of z
- Factor denominator into first-order factors
- Use partial fraction decomposition to get first-order terms

Example (con't)

$$\begin{array}{r} 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1\overline{\big)\,z^{-2}+2z^{-1}+1} \\ \underline{z^{-2}-3z^{-1}+2} \\ 5z^{-1}-1 \end{array}$$

$$X(z)=2+\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$

$$A_1=\left.\frac{1+2z^{-1}+z^{-2}}{1-z^{-1}}\right|_{z^{-1}=2}=\frac{1+4+4}{1-2}=-9$$

$$A_2=\left.\frac{1+2z^{-1}+z^{-2}}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=1}=\frac{1+2+1}{\frac{1}{2}}=8$$

- Find B_0 by polynomial division
- Express in terms of B_0
- Solve for A_1 and A_2

Example (con't)

- Express *X*(*z*) in terms of B_0, A_1, and A_2

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2} z^{-1}} + \frac{8}{1 - z^{-1}}$$

- Use table to obtain inverse *z*-transform

$$x[n] = 2\,\delta[n] - 9\left(\frac{1}{2}\right)^{n} u[n] + 8\,u[n]$$

- With the unilateral *z*-transform, or the bilateral *z*-transform with region of convergence, the inverse *z*-transform is unique

Z-transform Properties

- Linearity $a_1 x_1[n] + a_2 x_2[n] \Leftrightarrow a_1 X_1(z) + a_2 X_2(z)$
- Right shift (delay)

$$x[n-m]u[n-m] \Leftrightarrow z^{-m} X(z)$$

$$x[n-m]u[n] \Leftrightarrow z^{-m} X(z) + z^{-m}\left(\sum_{l=1}^{m} x[-l] z^{l}\right)$$

Second property used in solving difference equations

Second property derived in Appendix N of course reader by decomposing the left-hand side as follows:
x[*n*-*m*] *u*[*n*] = *x*[*n*-*m*] (*u*[*n*] – *u*[*n*-*m*]) + *x*[*n*-*m*] *u*[*n*-*m*]

Z-transform Properties

$$f_1[n]*f_2[n]=\sum_{m=-\infty}^{\infty}f_1[m]f_2[n-m]$$

- **Convolution definition**

$$Z\{f_1[n]*f_2[n]\}=Z\left\{\sum_{m=-\infty}^{\infty}f_1[m]f_2[n-m]\right\}$$

- **Take *z*-transform**

$$=\sum_{n=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty}f_1[m]f_2[n-m]\right)z^{-n}$$

- ***Z*-transform definition**

$$=\sum_{m=-\infty}^{\infty}f_1[m]\sum_{n=-\infty}^{\infty}f_2[n-m]z^{-n}$$

- **Interchange summation**

$$=\sum_{m=-\infty}^{\infty}f_1[m]\sum_{r=-\infty}^{\infty}f_2[r]z^{-(r+m)}$$

- **Substitute *r* = *n* - *m***

$$=\left(\sum_{m=-\infty}^{\infty}f_1[m]z^{-m}\right)\left(\sum_{r=-\infty}^{\infty}f_2[r]z^{-r}\right)$$

- ***Z*-transform definition**

$$=F_1(z)F_2(z)$$