Fourier Transform & Fourier Series

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Fourier Series

7.1 General Properties

Fourier series

A Fourier series may be defined as an expansion of a function in a series of sines and cosines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \qquad (7.1)$$

The coefficients are related to the periodic function f(x) by definite integrals: Eq.(7.11) and (7.12) to be mentioned later on.

The Dirichlet conditions:

(1) f(x) is a periodic function;

(2) f(x) has only a finite number of finite discontinuities;

(3) f(x) has only a finite number of extrem values, maxima and minima in the interval $[0,2\pi]$.

Fourier series are named in honor of Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Express *cos nx* and *sin nx* in exponential form, we may rewrite Eq.(7.1) as

$$\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right), \qquad \sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad (7.2)$$

in which

$$c_n = \frac{1}{2}(a_n - ib_n),$$

$$c_{-n} = \frac{1}{2}(a_n + ib_n), \qquad n > 0, \qquad (7.3)$$

and

$$c_0 = \frac{1}{2} a_0.$$

Completeness

One way to show the completeness of the Fourier series is to transform the trigonometric Fourier series into exponential form and compare It with a Laurent series.

If we expand f(z) in a Laurent series(assuming f(z) is analytic),

$$f(z) = \sum_{n=-\infty}^{\infty} d_n z^n. \qquad (7.4)$$

On the unit circle $z = e^{i\theta}$ and

$$f(z) = f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} d_n e^{in\theta}. \qquad (7.5)$$

The Laurent expansion on the unit circle has the same form as the complex Fourier series, which shows the equivalence between the two expansions. Since the Laurent series has the property of completeness, the Fourier series form a complete set. There is a significant limitation here. Laurent series cannot handle discontinuities such as a square wave or the sawtooth wave.

We can easily check the orthogonal relation for different values of the eigenvalue *n* by choosing the interval $[0,2\pi]$

$$\int_0^{2\pi} \sin mx \sin nx dx = \begin{cases} \pi\delta_{m,n}, & m \neq 0, \\ 0, & m = 0, \end{cases} \qquad (7.7)$$

$$\int_0^{2\pi} \cos mx \cos nx dx = \begin{cases} \pi\delta_{m,n}, & m \neq 0, \\ 2\pi, & m = n = 0, \end{cases} \qquad (7.8)$$

$$\int_0^{2\pi} \sin mx \cos nx dx = 0 \qquad \text{for all integer m and n.} \qquad (7.9)$$

By use of these orthogonality, we are able to obtain the coefficients

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

multipling $\cos mx$, and then integral from 0 to 2π

$$\int_0^{2\pi} \cos(mx) f(x) dx = \frac{a_0}{2} \int_0^{2\pi} \cos(mx) dx + \sum_{n=1}^{\infty} (a_n \int_0^{2\pi} \cos(nx)\cos(mx) dx + b_n \int_0^{2\pi} \sin(nx)\cos(mx) dx)$$

Similarly

$$\int_0^{2\pi} \sin(mx) f(x) dx = \frac{a_0}{2} \int_0^{2\pi} \sin(mx) dx + \sum_{n=1}^{\infty} (a_n \int_0^{2\pi} \cos(nx)\sin(mx) dx + b_n \int_0^{2\pi} \sin(nx)\sin(mx) dx)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt dt, \qquad (7.11)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt dt, \quad n = 0,1,2 \qquad (7.12)$$

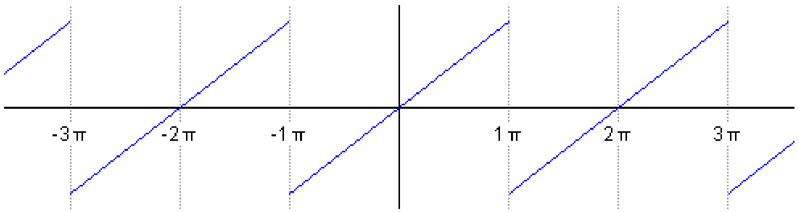
Substituting them into Eq.(7.1), we write

$$f(x)=\frac{1}{2\pi}\int_0^{2\pi} f(t)dt+\frac{1}{\pi}\sum_{n=1}^{\infty}(\cos nx\int_0^{2\pi} f(t)\cos ntdt+\sin nx\int_0^{2\pi} f(t)\sin ntdt)$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(t)dt+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} f(t)\cos n(t-x)dt, \qquad (7.13)$$

This equation offers one approach to the development of the Fourier integral and Fourier transforms.

Sawtooth wave



Let us consider a sawtooth wave

$$f(x)=\begin{cases} x, & 0\le x\le \pi \\ x-2\pi, & \pi\le x\le 2\pi. \end{cases} \qquad (7.14)$$

For convenience, we shall shift our interval from $[0,2\pi]$ to $[-\pi,\pi]$. In this interval we have simply f(x)=x. Using Eqs.(7.11) and (7.12), we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt = 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{2}{\pi}\int_{0}^{\pi} t\sin nt\,dt$$

$$= \frac{2}{\pi n}\left[-t\cos nt\Big|_0^{\pi} + \int_0^{\pi}\cos nt\,dt \right]$$

$$= \frac{2}{n}(-1)^{n+1},$$

So, the expansion of f(x) reads

$$f(x) = x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots + (-1)^{n+1}\frac{\sin nx}{n} + \cdots \right]. \qquad (7.15)$$

Figure 7.1 shows *f(x)* for the sum of 4, 6, and 10 terms of the series. Three features deserve comment.

1.There is a steady increase in the accuracy of the representation as the number of terms included is increased.

2.All the curves pass through the midpoint $f(x) = 0$ at $x = \pi$

7.2 ADVANTAGES, USES OF FOURIER SERIES

- Discontinuous Function

One of the advantages of a Fourier representation over some other representation, such as a Taylor series, is that it may represent a discontinuous function. An example id the sawtooth wave in the preceding section. Other examples are considered in Section 7.3 and in the exercises.

- Periodic Functions

Related to this advantage is the usefulness of a Fourier series representing a periodic functions . If *f(x)* has a period of 2π , perhaps it is only natural that we expand it in a series of functions with period 2π , $2\pi/2$, $2\pi/3$ $\cdots$ This guarantees that if our periodic f(x) is represented over one interval $[0,2\pi]$ or $[-\pi,\pi]$ the representation holds for all finite *x*.

At this point we may conveniently consider the properties of symmetry. Using the interval $[-\pi, \pi]$, $\sin x$ is odd and $\cos x$ is an even function of *x*. Hence , by Eqs. (7.11) and (7.12), if *f(x)* is odd, all $a_n = 0$ if *f(x)* is even all $b_n = 0$. In other words,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad f(x) \quad \text{enen,} \qquad (7.21)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad f(x) \quad \text{odd.} \qquad (7.21)$$

Frequently these properties are helpful in expanding a given function.

We have noted that the Fourier series periodic. This is important in considering whether Eq. (7.1) holds outside the initial interval. Suppose we are given only that

$$f(x) = x, \qquad 0 \le x < \pi \qquad (7.23)$$

and are asked to represent *f(x)* by a series expansion. Let us take three of the infinite number of possible expansions.

Continuous Fourier Transform (FT)

- Transforms a signal (i.e., function) from the spatial (x) domain to the frequency (u) domain.

Forward FT: $$F(f(x)) = F(u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}dx$$

Inverse FT: $$F^{-1}(F(u)) = f(x) = \int_{-\infty}^{\infty} F(u)e^{j2\pi ux}du$$

where $$e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$$

Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

Arbitrary function ⟶ Single Analytic Expression

Spatial Domain (*x*) ⟶ Frequency Domain (*u*)

(Frequency Spectrum *F(u)*)

Inverse Fourier Transform (IFT)

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} dx$$

Fourier Transform

- Also, defined as:

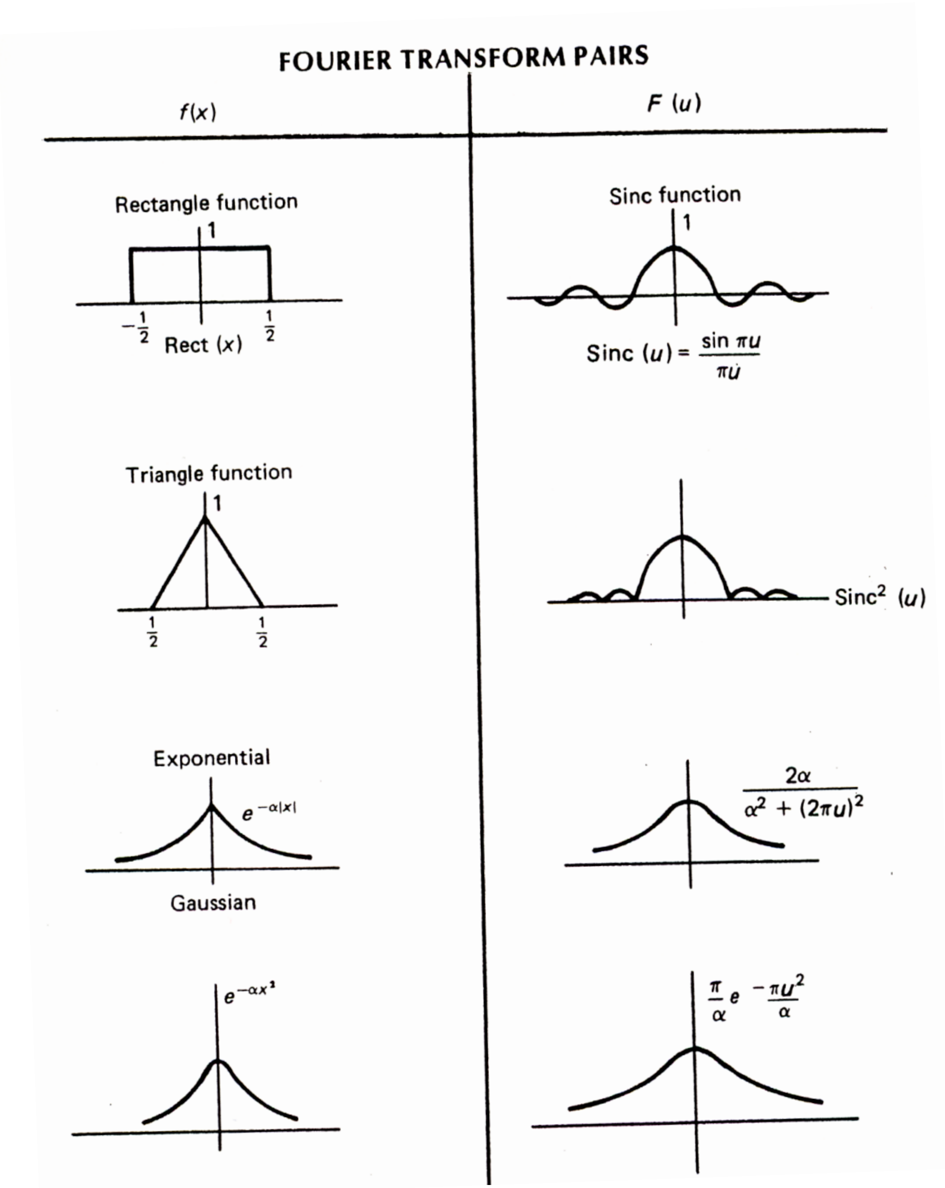
$$F(u)=\int_{-\infty}^{\infty} f(x)e^{-iux}dx$$

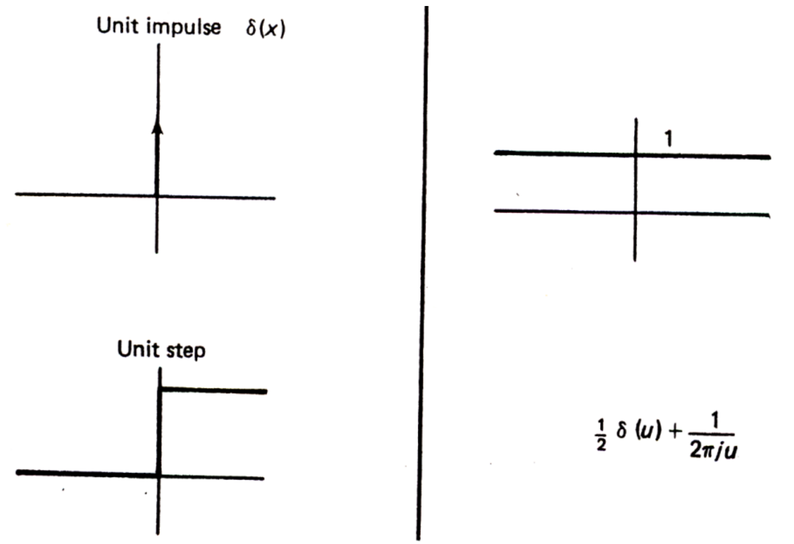
Note: $e^{ik}=\cos k+i\sin k \qquad i=\sqrt{-1}$

- Inverse Fourier Transform (IFT)

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{iux}dx$$

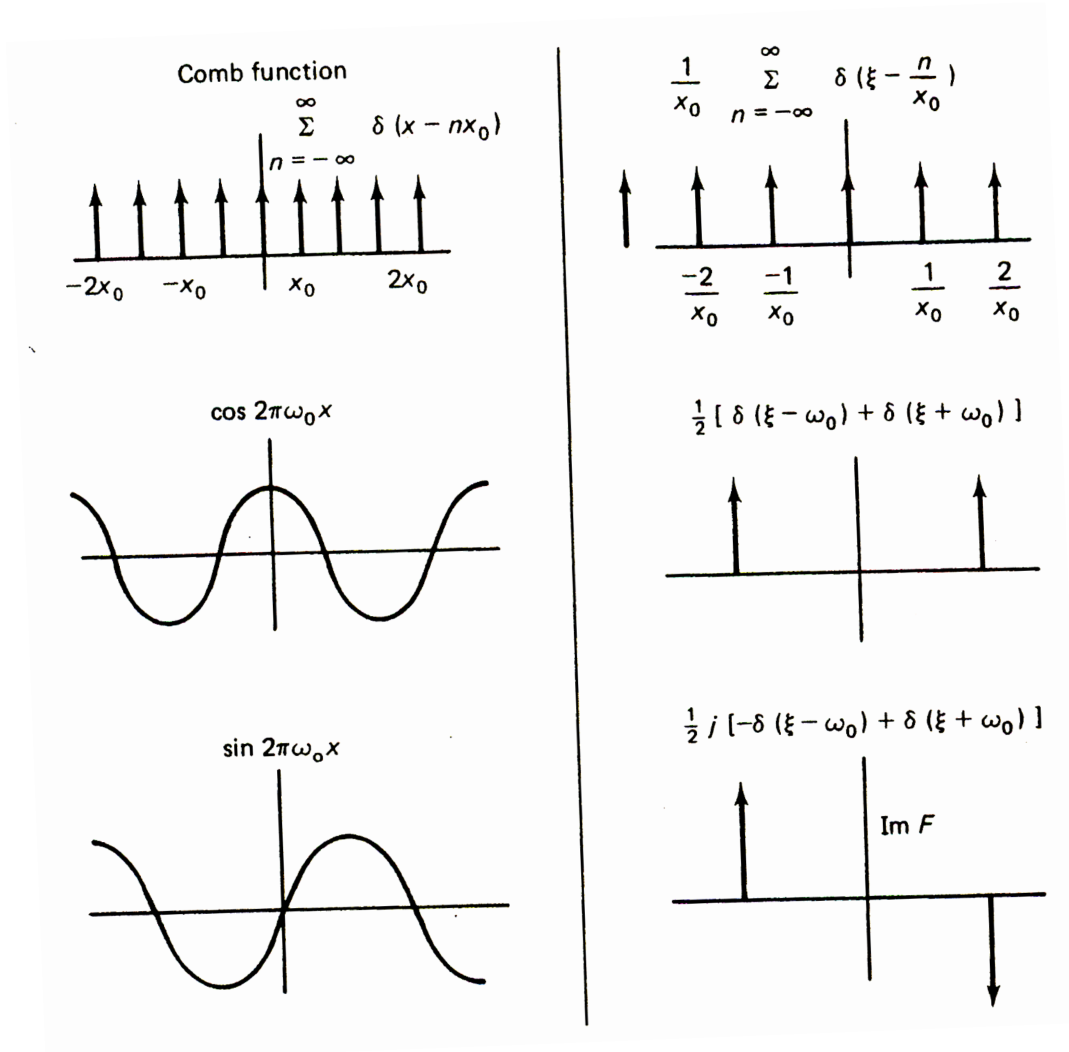
Fourier Transform Pairs (I)





Note that these are derived using angular frequency ($e^{-i\omega x}$)

Fourier Transform Pairs (I)



Note that these are derived using angular frequency (e^{-iux})

Properties of Fourier Transform

	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency ($e^{-i2\pi ux}$)

Properties of Fourier Transform

Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\xi)|^2 d\xi$$

$$\int_{-\infty}^{\infty} f(x) g^*(x)\, dx = \int_{-\infty}^{\infty} F(\xi) G^*(\xi)\, d\xi$$

$f(x)$	$F(\xi)$
Real(R)	Real part even (RE) Imaginary part odd (IO)
Imaginary (I)	RO,IE
RE,IO	R
RE,IE	I
RE	RE
RO	IO
IE	IE
IO	RO
Complex even (CE)	CE
CO	CO